Constraint Handling in Multiobjective Optimization

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Contents

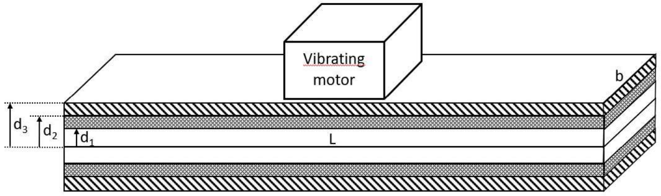
Introduction

Background (Liang et al., 2023)

- Optimization problems often include both **multiple objectives and constraints**
- **Multiobjective evolutionary algorithms (MOEAs) –** a natural extension of EAs for solving **multiobjective optimization problems (MOPs)**
- Dealing with **constrained multiobjective optimization problems (CMOPs)** long ignored – believed that **constraint handling techniques (CHTs)** for single-objective problems can easily be incorporated into MOEAs
- Recent shift of research focus towards CMOPs

Motivating Example (i)

Vibrating platform (Messac, 1996)



- Engineering design problem
- Design variables: d_1, d_2, d_3, b, L
- Task: maximize the fundamental frequency of the platform, minimize its cost

Motivating Example (ii)

Objectives

- f_1 ... fundamental frequency

$$f_1(d_1, d_2, d_3, b, L) = \frac{\pi}{2L^2}\left(\frac{EI}{\mu}\right)^{1/2}$$

$$EI = \frac{2b}{3}\left[E_1 d_1^3 + E_2(d_2^3 - d_1^3) + E_3(d_3^3 - d_2^3)\right]$$

$$\mu = 2b\left[\rho_1 d_1 + \rho_2(d_2 - d_1) + \rho_3(d_3 - d_2)\right]$$

- f_2 ... cost

$$f_2(d_1, d_2, d_3, b) = 2b\left[c_1 d_1 + c_2(d_2 - d_1) + c_3(d_3 - d_2)\right]$$

Motivating Example (iii)

Constraints

- Boundary constraints

$$0.01 \leq d_1 \leq 0.6$$
$$0.01 \leq d_2 \leq 0.6$$
$$0.01 \leq d_3 \leq 0.6$$
$$0.35 \leq b \leq 0.5$$
$$3 \leq L \leq 6$$

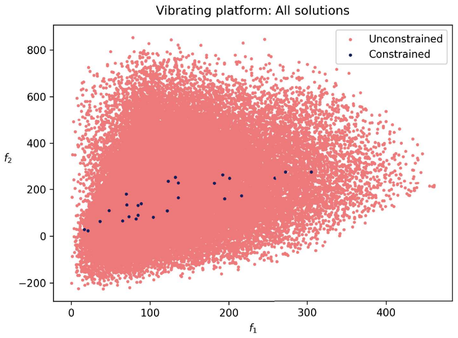
- Inequality constraints

$$0 \leq d_2 - d_1 \leq 0.01$$
$$0 \leq d_3 - d_2 \leq 0.01$$
$$\mu L \leq 2800$$

Motivating Example (iv)

Some problem characteristics

- 5 design variables
- 2 objectives
- 5 constraints
- feasibility ratio* $< 10^{-5}$



*Estimated empirically through solution sampling. Denotes the proportion of feasible solutions among the sampled solutions.

Challenges of Constrained Multiobjective Optimization

- Need to handle both objectives and constraints
- Feasibility ratio can be low
- Objectives and constraints may or may not be correlated
- Feasible region can be disconnected
- etc.

Prerequisites: CMOP Formulation

Constrained multiobjective optimization problem (CMOP):

$$\begin{aligned} \text{minimize} \quad & f_m(x), \quad m = 1, \dots, M \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \dots, I \\ & h_i(x) = 0, \quad i = I+1, \dots, I+J \end{aligned}$$

where

- $x = (x_1, \dots, x_n)$ ... decision vector
- $S \subseteq \mathbb{R}^n$ ... decision space
- $f_m : S \to \mathbb{R}$ ... objective functions
- $g_i : S \to \mathbb{R}$ ... inequality constraint functions
- $h_i : S \to \mathbb{R}$ ... equality constraint functions

Prerequisites: Constraint Violation

The equality constraints are usually reformulated into inequality constraints:

$$g_i(x) = |h_i(x)| - \epsilon \leq 0, \quad i = I+1, \dots, I+J$$

where $\epsilon > 0$ is a user-defined tolerance value (e.g. 10^{-4})

Constraint violation for a single constraint:

$$v_i(x) = \max(g_i(x), 0)$$

Overall constraint violation for all constraints combined:

$$v(x) = \sum_{i=1}^{I+J} v_i(x)$$

Constraint Handling Techniques (CHTs)

CHTs for Single-Objective Optimization

- Penalty functions
- Solution repair
- Separation of objectives and constraints
- Other approaches

Penalty Functions (i)

Idea: transform a constrained problem into an unconstrained one by adding penalty terms to the objective function:

$$f'(x) = f(x) + \sum_{i=1}^{I+J} p_i \cdot v_i(x)$$

where

- $f'(x)$ ... modified objective function
- p_i ... penalty factors

Penalty Functions (ii)

Variants

- Death penalty
- Static penalty
- Dynamic penalty
- Adaptive penalty
- Adjustments and modifications of these variants

Penalty Functions (iii)

- Most popular CHT
- Issue: Setting the penalty factors
- Penalties too low: The algorithm spends a lot of time exploring the infeasible region
- Penalties too high: The algorithm may have difficulties detecting the optimum when it is located at the border of the feasible region

Solution Repair

- Idea: Introduce a procedure for converting infeasible solutions to feasible ones
- Repaired solutions can be used for evaluation only, or can replace the original solutions in the population (Lamarckian evolution)
- Problem-dependent, a specific procedure needed for each problem
- Suitable when repair is easy and of low computational cost

Separation of Objectives and Constraints

- In contrast to penalty functions, these techniques handle objectives and constraints separately
- Examples:
  - Superiority of feasible solutions: Always assign a higher fitness to feasible solutions than to infeasible ones
  - Multiobjective optimization approach: $K+1$ objectives where K is the number of constraints
  - Coevolution: evolve two interacting populations

Other Approaches

- Special representations and operators
- Hybrid techniques
- Ensembles of CHTs
- Landscape-aware constraint handling: Using the concept of violation landscape (Malan, 2018; Malan and Moser, 2019)

CHTs for Multiobjective Optimization

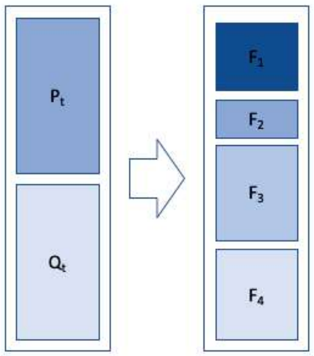
- CHTs incorporated in Nondominated sorting genetic algorithm II (NSGA-II)
- CHTs incorporated in Multiobjective evolutionary algorithm based on decomposition (MOEA/D)
- Advanced techniques

CHTs incorporated in NSGA-II

- Constrained dominance principle (CDP)
- Stochastic ranking (SR)
- Penalty function

NSGA-II

- **The most frequently used algorithm in constrained multiobjective optimization**
- CHT usually incorporated within the sorting procedure



NSGA-II: CDP (Deb, Pratap, Agarwal, et al., 2002)

A solution *x* is said to **constrained-dominate** a solution *y*, if any of the following conditions is true:

- Solution *x* is feasible and solution *y* is not
- Solutions *x* and *y* are both infeasible, but solution *x* has a smaller overall constraint violation than *y*
- Solutions *x* and *y* are feasible and solution *x* dominates solution *y*

The most commonly used CHT in constrained multiobjective optimization (Z. Ma et al., 2019)

NSGA-II: SR (Geng et al., 2006)

Stochastic ranking selection:

- Feasible solutions are compared based on the dominance relation
- Infeasible solutions are compared either based on on the overall constraint violation or dominance relation
- **The comparison criterion is randomly selected**

IS-MOEA:

- Based on NSGA-II and stochastic ranking selection
- Uses the **infeasible elitists preservation**

NSGA-II: Penalty Function (Woldesenbet et al., 2009)

Transform the objective functions into:

$$f_i'(x) = \begin{cases} f_i(x) & \text{if } x \text{ is feasible} \\ v(x) & \text{if } x \text{ is infeasible and } \rho_F(P) = 0 \\ p_i(x) + d_i(x) & \text{if } x \text{ is infeasible and } \rho_F(P) \neq 0 \end{cases}$$

where

$$p_i(x) = (1 - \rho_F(P))v(x) + \rho_F(P)f_i(x)$$

and

$$d_i(x) = \sqrt{f_i(x)^2 + v(x)^2}$$

CHTs incorporated in MOEA/D

- CDP
- SR
- ϵ-constraint (Epsilon)
- Improved ϵ-constraint (IEpsilon)

MOEA/D

The original problem is decomposed into multiple subproblems

The **Tchebycheff aggregation function** is the most widely used decomposition approach in constrained multiobjective optimization

A subproblem is defined as follows:

$$\text{minimize } g(x \mid \lambda, z^*) = \max_{1 \leq i \leq m} \{\lambda_i \, |f_i(x) - z_i^*|\}$$

where z^* is an approximation for the ideal point and λ a weight vector

Idea: The aggregation function can be seen as a fitness of the subproblem $\rightarrow$ Easy to incorporate CHTs for single-objective optimization

MOEA/D-DE (i)

MOEA/D-DE:

- Employs **differential evolution** (DE) operator for generating new solutions
- Limits the maximal number of solutions replaced by a better child solution, n_r

The most interesting part of MOEA/D-DE is the **update phase**

Algorithm 1: Update neighboring solutions N with x

```
c ← 0;
while c < n_r and N ≠ ∅ do
    randomly pick y ∈ N;
    if g(x) < g(y) then
        y ← x, c ← c + 1;
    end
    N ← N − {y};
end
```


MOEA/D-DE (ii)

Algorithm 2: Update neighboring solutions N with x

```
c ← 0;
while c < n_r and N ≠ ∅ do
    randomly pick y ∈ N;
    if g(x) < g(y) then
        y ← x, c ← c + 1;
    end
    N ← N − {y};
end
```

Algorithm 3: Update neighboring solutions N with x

```
c ← 0;
while c < n_r and N ≠ ∅ do
    randomly pick y ∈ N;
    if x ⪯ y then
        y ← x, c ← c + 1;
    end
    N ← N − {y};
end
```


MOEA/D: CDP and SR (Jan et al., 2013)

MOEA/D-CDP:

$$x \preceq_{\mathrm{CDP}} y \Leftrightarrow \begin{cases} g(x) < g(y) & \text{if } v(x) = v(y) = 0 \\ v(x) < v(y) & \text{otherwise} \end{cases}$$

MOEA/D-SR:

$$x \preceq_{\mathrm{SR}} y \Leftrightarrow \begin{cases} g(x) < g(y) & \text{if } v(x) = v(y) = 0 \text{ or rand} < p \\ v(x) < v(y) & \text{otherwise} \end{cases}$$

MOEA/D: ϵ-Constraint Technique (Asafuddoula et al., 2012)

MOEA/D-Epsilon:

$$x \preceq_{\epsilon} y \Leftrightarrow \begin{cases} g(x) < g(y) & \text{if } v(x) = v(y) \text{ or } (v(x) \leq \epsilon \text{ and } v(y) \leq \epsilon) \\ v(x) < v(y) & \text{otherwise} \end{cases}$$

The ϵ value is updated in each generation:

$$\epsilon = \overline{v} \cdot \rho_F(P)$$

where

$$\overline{v} = \frac{1}{|P|} \sum_{x \in P} v(x)$$

MOEA/D: Improved ϵ-Constraint Technique (Fan, W. Li, Cai, Huang, et al., 2019)

MOEA/D-IEpsilon:

The ϵ value is updated in each generation:

$$\epsilon(t) = \begin{cases} v(x^\theta) & \text{if } t = 0 \\ (1-\tau)\epsilon(t-1) & \text{if } \rho_F(P) < \alpha \text{ and } t < T_c \\ (1+\tau)v_{\max} & \text{if } \rho_F(P) \geq \alpha \text{ and } t < T_c \\ 0 & \text{if } t \geq T_c \end{cases}$$

where τ, α, T_c are user-defined parameters and $v(x^\theta)$ is the overall constraint violation of the top θ-th individual in the initial population

Advanced Techniques

- Ensembles
- Multiple phase techniques
- Multiple population techniques
- Hybrids
- Coevolution

Advanced Techniques: Two-Phase Framework (Z. Liu and Y. Wang, 2019)

Two-phase framework:

1. **First phase:** Solve a constrained single-objective problem

$$\begin{aligned} \text{minimize} \quad & f'(x) = \sum_{i=1}^{M} f_i(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \ldots, I + J \end{aligned}$$

2. **Second phase:** Apply constrained multiobjective optimization on the original problem **starting with solutions obtained in the first phase**

ToP:

- Differential evolution in the first phase
- NSGA-II (CDP) or IDEA in the second phase

Advanced Techniques: Push and Pull Search (Fan, W. Li, Cai, H. Li, et al., 2019b)

Search is divided into **two stages**:

1. **Push** ignores constraints
2. **Pull** handles infeasible solutions

PPS-MOEA/D:

- Push stage: MOEA/D-DE
- Pull stage: MOEA/D-IEpsilon
- **Parameters for the pull stage assessed in the push stage**

Advanced Techniques: Two-Archive Evolutionary Algorithm (K. Li et al., 2019)

Two complementary archives:

- **Convergence archive:** Maintain the convergence and feasibility of the evolution process
- **Diversity archive:** Maintain the diversity of the evolution process
- A restricted mating mechanism combines parents from the two archives

C-TAEA:

- Based on MOEA/D and M2M framework (decomposition of the original multiobjective optimization problem into multiple simpler subproblems)

Advanced Techniques: Coevolutionary Framework (Tian et al., 2021)

Two populations:

1. One population is solving the **original problem**
2. The other one is solving a **helper problem**—a simpler problem derived from the original one

CCMO:

- Coevolutionary framework incorporated into NSGA-II
- Helper problem: original problem without constraints

Advanced Techniques: Additional (i)

- **MSCMO**: Multi-stage evolutionary algorithm for constrained multiobjective optimization (H. Ma et al., 2021)
- **POCEA**: Paired offspring generation-based evolutionary algorithm (He et al., 2021)
- **TriP**: Tri-population based coevolutionary framework (Ming et al., 2022)
- **TSTI**: Two stage evolutionary algorithm based on three indicators (Dong et al., 2022)

Advanced Techniques: Additional (ii)

- **TPEA**: Constrained multiobjective optimization with escape and expansion forces (Z. Liu, Wu, et al., 2023)
- **MTCMO**: Dynamic auxiliary task-based evolutionary multitasking for constrained multiobjective optimization (Qiao et al., 2023)
- **COEA-DAS**: A coevolutionary algorithm with detection and supervision strategies for constrained multiobjective optimization (S. Liu et al., 2024)

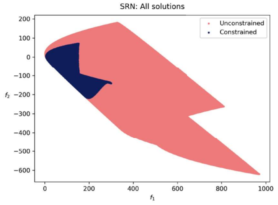
Test Problems

Overview

- Artificial test problems
- Artificial test suites
- Real-world test problems based on mathematical models
- Real-world test problems based on simulation

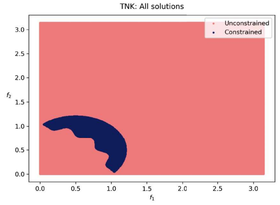
Artificial Test Problems: SRN (Srinivas et al., 1995)

$$
\begin{aligned}
\text{min. } & f_1(x) = 2 + (x_1 - 2)^2 + (x_2 - 1)^2 \\
\text{min. } & f_2(x) = 9x_1 - (x_2 - 1)^2 \\
\text{s.t. } & g_1(x) = x_1^2 + x_2^2 - 225 \leq 0 \\
& g_2(x) = x_1 - 3x_2 + 10 \leq 0 \\
& x_1, x_2 \in [-20, 20]
\end{aligned}
$$



Artificial Test Problems: TNK (Tanaka et al., 1995)

$$
\begin{aligned}
\text{min. } & f_1(x) = x_1 \\
\text{min. } & f_2(x) = x_2 \\
\text{s.t. } & g_1(x) = x_1^2 + x_2^2 - 1 - 0.1\cos(16\tan^{-1}\frac{x}{y}) \geq 0 \\
& g_2(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.5 \leq 0 \\
& x_1, x_2 \in [0, \pi]
\end{aligned}
$$

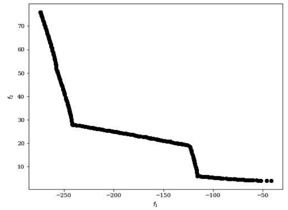


Artificial Test Problems: OSY (Osyczka et al., 1995)

$$\max.\ f_1(x) = 25(x_1 - 2)^2 + (x_2 - 2)^2 + (x_3 - 1)^2 + (x_4 - 4)^2 + (x_5 - 1)^2$$

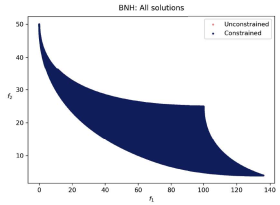
$$\min.\ f_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$$

$$\text{s.t. } g_1(x) = x_1 + x_2 - 2 \geq 0$$
$$g_2(x) = 6 - x_1 - x_2 \geq 0$$
$$g_3(x) = 2 - x_2 + x_1 \geq 0$$
$$g_4(x) = 2 - x_1 - 3x_2 \geq 0$$
$$g_5(x) = 4 - (x_3 - 3)^2 - x_4 \geq 0$$
$$g_6(x) = (x_5 - 3)^2 + x_6 - 4 \geq 0$$
$$x_1, x_2, x_6 \in [0, 10], x_3, x_5 \in [1, 5], x_4 \in [0, 6]$$



Artificial Test Problems: BNH (Binh et al., 1997)

$$\min.\ f_1(x) = 4(x_1^2 + x_2^2)$$
$$\min.\ f_2(x) = (x_1 - 5)^2 + (x_2 - 5)^2$$
$$\text{s.t. } g_1(x) = (x_1 - 5)^2 + x_2^2 - 25 \leq 0$$
$$g_2(x) = (x_1 - 8)^2 - (x_2 + 3)^2 - 7.7 \geq 0$$
$$x_1 \in [0, 5], x_2 \in [0, 3]$$



Artificial Test Problems: Issues with SRN, TNK, OSY, BNH

Issues:

- Low dimensionality
- Not hard to solve
- Complexity/difficulty not tunable

→ Further proposals: Frameworks for constructing harder tunable problems

Artificial Test Suites: CTP (Deb, Pratap, and Meyarivan, 2001)

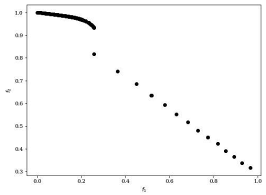
Constrained test problems (CTPs)

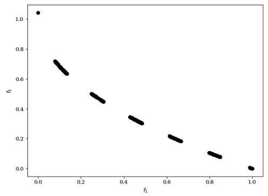
Scalable number of decision variables and **tunable constraint difficulties**

Two kinds of difficulty:

- Difficulty in the vicinity of PF
- Difficulty in the entire search space

8 bi-objective CMOPs including 1–2 constraints

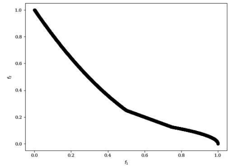


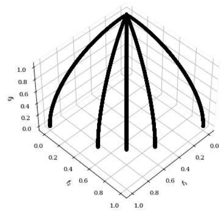


Artificial Test Suites: CF (Zhang et al., 2008)

Constrained multiobjective test problems from the CEC 2009 Special Session and Competition (CFs)

10 problems with 2 or 3 objectives and 1 or 2 constraints





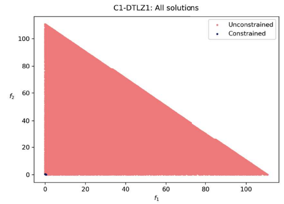
Artificial Test Suites: C-DTLZ (Jain et al., 2014)

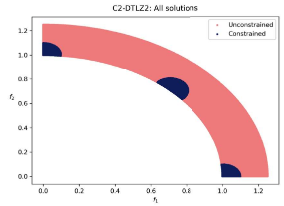
Constrained DTLZ problems (C-DTLZs)

Three types of CMOPs:

- C1: unconstrained PF still optimal, barrier in approaching PF
- C2: only parts of unconstrained PF feasible
- C3: unconstrained PF no longer optimal

6 scalable CMOPs in the number of objectives and constraints





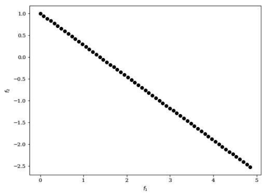
Artificial Test Suites: NCTP (J. Li et al., 2016)

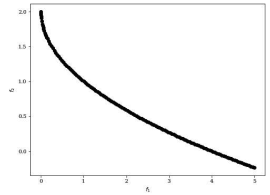
New constrained test problems (NCTPs)

An extension of CTPs:

- Difficulty of convergence is increased
- Infeasible region is increased by an additional constraint

18 bi-objective CMOPs with 1 or 2 constraints





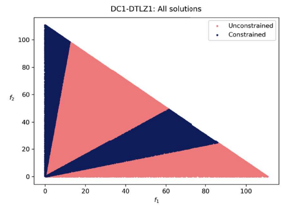
Artificial Test Suites: DC-DTLZ (K. Li et al., 2019)

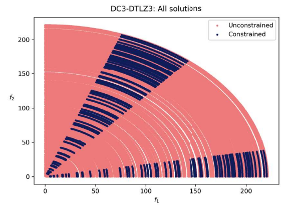
Constrained DTLZ problems where constraints act in the decision space (DC-DTLZs)

Three types of constraints:

- DC1: several infeasible segments
- DC2: unconstrained PF still optimal, barrier in approaching PF
- DC3: decision space consists of several feasible regions

6 scalable CMOPs in the number of objectives and constraints





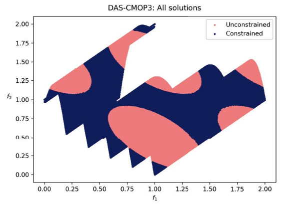
Artificial Test Suites: DAS-CMOP (Fan, W. Li, Cai, H. Li, et al., 2019a)

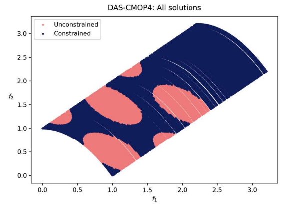
Difficulty-adjustable and scalable CMOPs (DAS-CMOPs)

Test problem kit considering basic difficulty types:

- T1: diversity hardness
- T2: feasibility hardness
- T3: convergence hardness

9 CMOPs of increasing hardness, scalable in the number of objectives

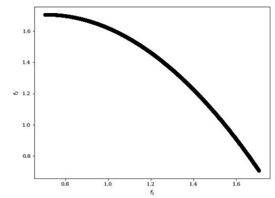


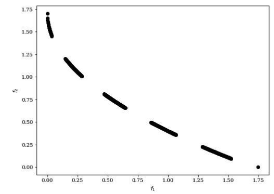


Artificial Test Suites: LIR-CMOP (Fan, W. Li, Cai, Huang, et al., 2019)

Large infeasible region CMOPs (LIR-CMOPs)

14 CMOPs with 2 or 3 objectives and 2 or 3 constraints





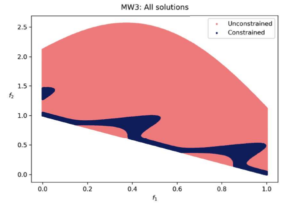
Artificial Test Suites: MW (Z. Ma et al., 2019)

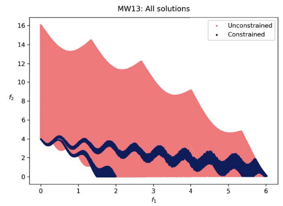
Ma and Wang problems (MWs)

Four types of CMOPs:

- Type I: unconstrained PF remains feasible
- Type II: constrained PF is a part of the unconstrained PF
- Type III: constrained PF consists of a part of the unconstrained PF and part of a boundary
- Type IV: unconstrained PF no longer optimal

11 bi-objective CMOPs and 3 scalable in the number of objective with 1–4 constraints





Artificial Test Suites: Others

- **DOC**: Constrained multiobjective optimization problems with constraints in the decision and objective space (Z. Liu and Y. Wang, 2019)
- **Eq-DTLZ** and **Eq-IDTLZ**: Benchmark for equality constrained multiobjective optimization (Cuate et al., 2020)
- **CLSMOP**: Constrained large-scale mutliobjective optimization problems (He et al., 2021)
- **ZXH-CF**: Constrained Multiobjective Optimization: Test Problem Construction and Performance Evaluations (Zhou et al., 2021)

Real-World Test Problems Based on Mathematical Models (i) (Tanabe and Ishibuchi, 2020)

Constrained real-world problem suite (CRE)

A collection of real-world test problems based on mathematical models:

- Mechanical design problems

8 problems with 3–7 variables, 2–5 objectives, and 1–11 constraints

Real-World Test Problems Based on Mathematical Models (ii)

Real-world constrained multiobjective optimization problems (RCMs) from CEC 2021 Special Session and Competition and GECCO 2021 Competition[1]

A collection of real-world test problems based on mathematical models:

- Mechanical design problems
- Chemical engineering optimization problems
- Process synthesis optimization problems
- Power systems optimization problems

50 problems with 2–34 variables, 2–5 objectives, and 1–29 constraints

[1]https://www3.ntu.edu.sg/home/epnsugan/index_files/CEC2021/CEC2021-1.htm

Real-World Test Problems Based on Simulations (i)

Mazda benchmark problem[2]:

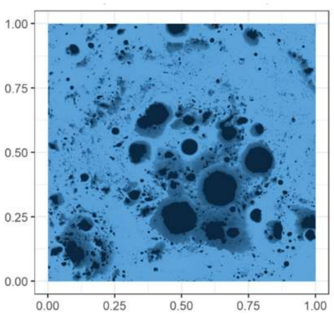
- Based on a real-world car structure design
- 222 design variables
- 2 objectives:
  - Minimize the total weight of three car models
  - Maximize the number of their common parts
- 54 constraints

[2]http://ladse.eng.isas.jaxa.jp/benchmark/

Real-World Test Problems Based on Simulations (ii)

Lunar lander landing site selection[3]:

- 2 design variables: coordinates x, y
- 3 objectives:
  - Total communication time
  - Continuous shade days
  - Landing point inclination angle
- 2 constraints
  - Max. continuous shade days
  - Max. landing point inclination angle



[3]http://www.jpnsec.org/files/competition2018/EC-Symposium-2018-Competition-English.html

Real-World Test Problems Based on Simulations (iii)

Wind turbine design problem[4]:

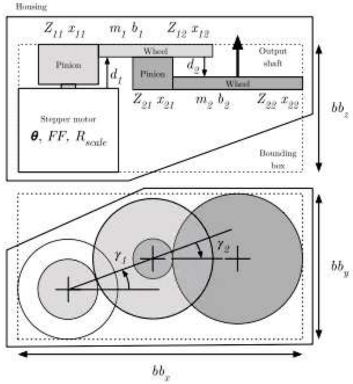
- Based on a real-world wind turbine design
- 32 design variables
- 5 objectives:
  - Annual power production
  - Average annual cost
  - Tower base load
  - Blade tip speed
  - Fatigue damage
- 22 constraints

[4] http://www.jpnsec.org/files/competition2019/EC-Symposium-2019-Competition-English.html

Real-World Test Problems Based on Simulations (iv) (Picard et al., 2021)

Multiobjective design of actuators (MODAct):

- Design of electro-mechanical actuators
- 20 CMOPs with 20 design variables
- 2–5 objectives:
  - Total cost (min.)
  - Torque excess (max.)
  - Harmonic mean of safety factors (max.)
  - Elec. to mech. energy conversion (max.)
  - Transmission ratio (min.)
- 7–10 constraints



Algorithm Performance Assessment

Performance Indicators

Any popular performance indicator for multiobjective optimization can be adapted for CMOPs by removing infeasible solutions

The most frequently used indicators in the literature are:

- Hypervolume (HV)
- Generational distance (GD and GD^{+})
- Inverted generational distance (IGD and IGD^{+})
- Epsilon indicator (EPS)

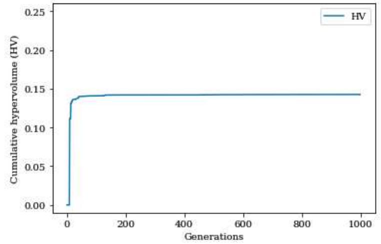
It is very important to use **archives**, **cumulative indicators** or **empirical runtime distributions**

Constraint-Related Measures

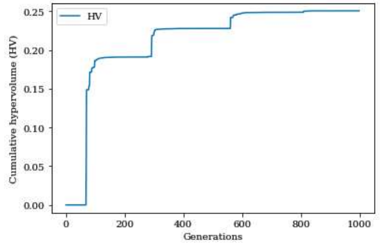
- Minimum of overall constraint violations
- Mean of overall constraint violations
- Feasibility ratio

Example: Hypervolume (LIR-CMOP1)

MOEA/D-CDP

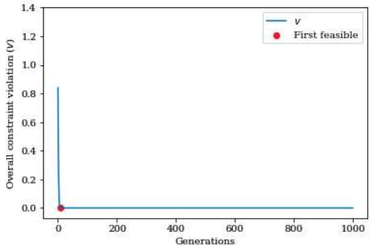


MOEA/D-IEpsilon

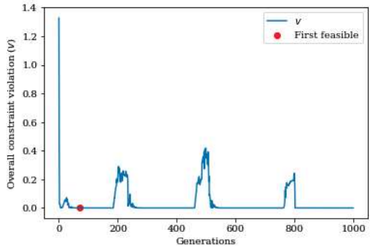


Example: Overall Constraint Violation (LIR-CMOP1)

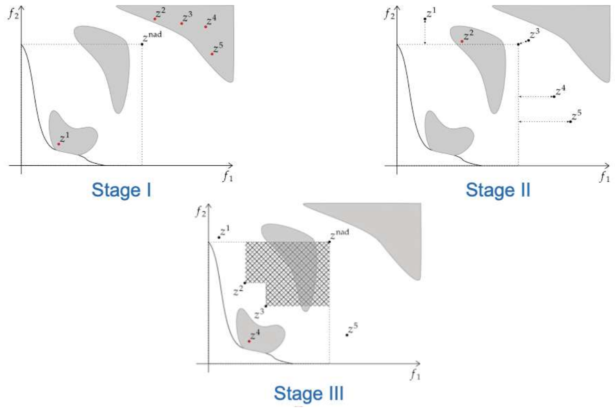
MOEA/D-CDP



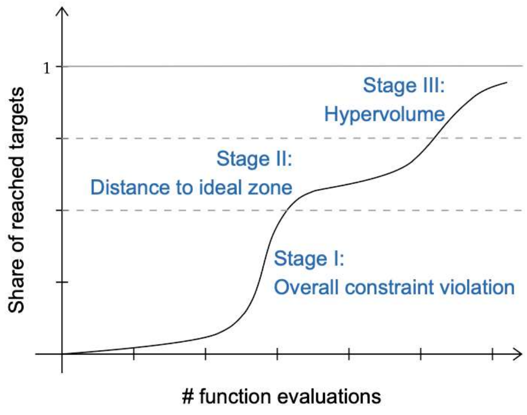
MOEA/D-IEpsilon



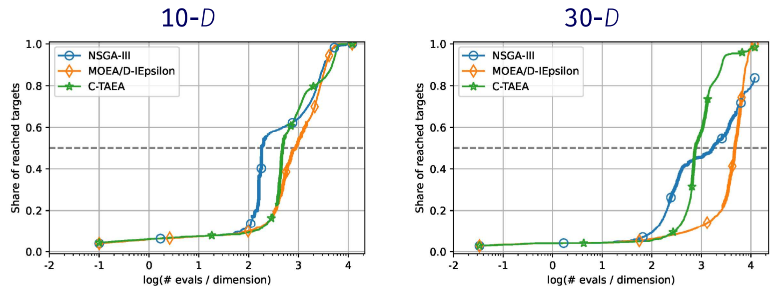
Three Stages of the Search (Vodopija et al., 2024)



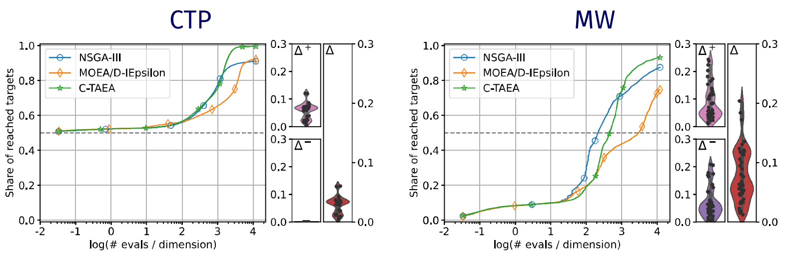
Empirical Runtime Distribution



Example: MW12



Example: CTP vs. MW

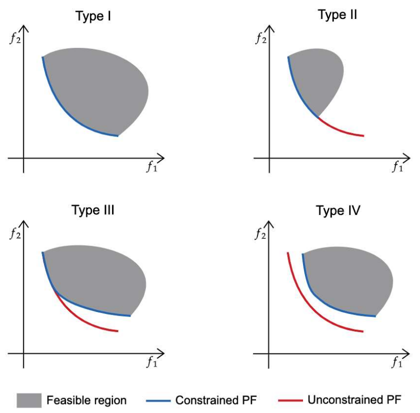


Problem Characterization

Overview

- Type of CMOPs
- Pareto front shapes
- Problem landscapes

Type of CMOPs (Z. Ma et al., 2019)

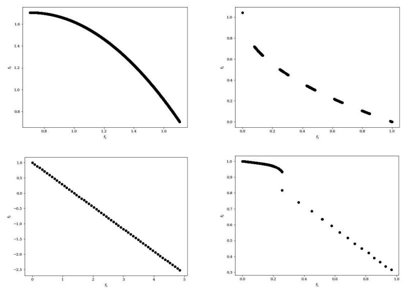


Test Suite Comparison: Type of CMOPs

Type	CTP	CF	C-DTLZ	NCTP	DC-DTLZ	DAS-CMOP	LIR-CMOP	MW
I	✓		✓	✓	✓	✓	✓	✓
II	✓	✓	✓		✓	✓	✓	✓
III		✓		✓		✓	✓	✓
IV	✓		✓	✓			✓	✓

Pareto Front Shapes

- Linear / Convex / Concave
- Connected / Disconnected / Discrete
- Mixed



Test Suite Comparison: Pareto Front Shapes

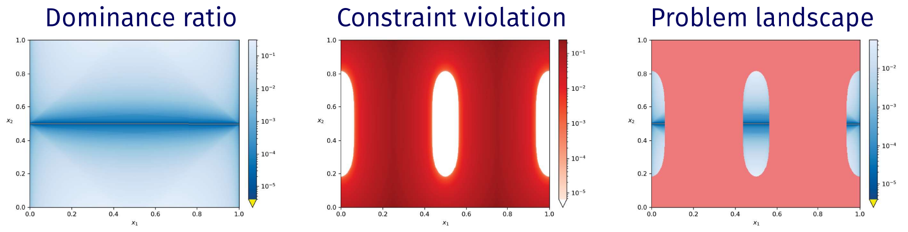
Type	CTP	CF	C-DTLZ	NCTP	DC-DTLZ	DAS-CMOP	LIR-CMOP	MW
Linear	✓	✓	✓	✓	✓	✓		✓
Convex	✓	✓	✓	✓		✓	✓	
Concave		✓	✓		✓	✓	✓	✓
Conn.	✓		✓	✓	✓	✓	✓	✓
Disconn.	✓	✓	✓		✓	✓	✓	✓
Discrete	✓	✓		✓		✓	✓	✓
Mixed	✓	✓		✓		✓	✓	✓

Problem Landscape

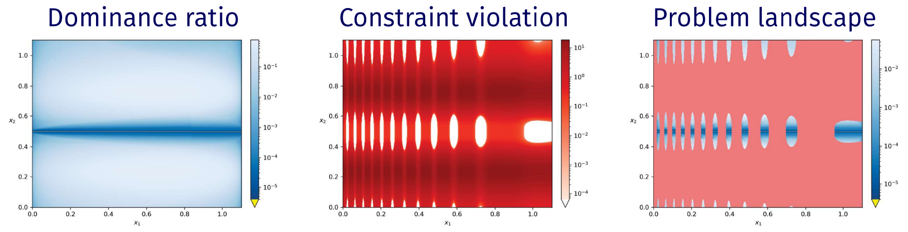
Constrained multiobjective problem landscape, $\mathcal{L}(S, f, v, d)$:

- $S \subseteq \mathbb{R}^n$ ... decision space
- $f : S \to \mathbb{R}^M$ ... objective vector function
- $v : S \to \mathbb{R}$ ... overall constraint violation function
- $d : S \times S \to \mathbb{R}$ ... distance metric

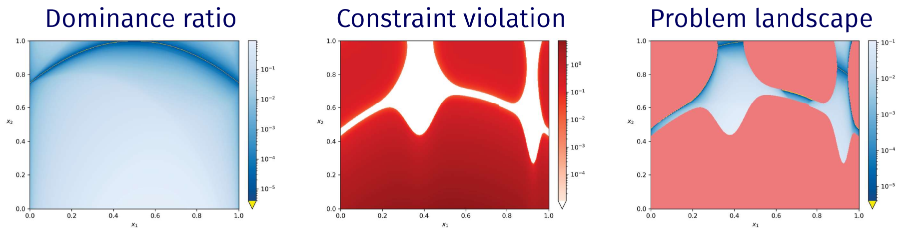
Example: C2-DTLZ2



Example: MW6



Example: MW7



Exploratory Landscape Analysis: State of the Art (i)

Multiobjective optimization:

- Limited studies in the combinatorial (Verel et al., 2013; Daolio et al., 2017; Liefooghe, Daolio, et al., 2020) and continuous context (Liefooghe, Verel, et al., 2021)
- Initial attempts to **visualize bi-objective continuous problems** (Fonseca, 1995; Kerschke, H. Wang, et al., 2016; Kerschke and Grimme, 2017; Schäpermeier et al., 2021)

Constrained single-objective optimization:

- Preliminary study on the **characterization of constrained single-objective optimization problems** (Malan, Oberholzer, et al., 2015)
- Incorporation of these characteristics to **guide the constraint handling** (Malan, 2018; Malan and Moser, 2019)

Exploratory Landscape Analysis: State of the Art (ii)

Constrained multiobjective optimization:

- Three preliminary studies on exploratory landscape analysis exist in the literature (Picard et al., 2021; Vodopija et al., 2022; Alsouly et al., 2023)

Exploratory Landscape Analysis (i) (Picard et al., 2021)

Goals:

- Analyze the **effect of constraints** on search and objective spaces
- Measure the **feasibility ratio**
- Quantify the **relationship between objectives and constraints**
- Measure the **disjointedness of feasible regions**

Methods:

- Uniform sampling
- Random walk

Exploratory Landscape Analysis (ii) (Alsouly et al., 2023)

Goals:

- Explore the **relationship between MOEA performance and CMOP characteristics**
- Analyze the *y*-**distribution of problem landscapes**
- Analyze the **interaction between constraints and objectives**
- Measure the **ruggedness** of problem landscapes
- Explore the **connectedness of Pareto fronts and sets**

Methods:

- Uniform sampling
- Random walk

Exploratory Landscape Analysis (iii) (Vodopija et al., 2022)

Goals:

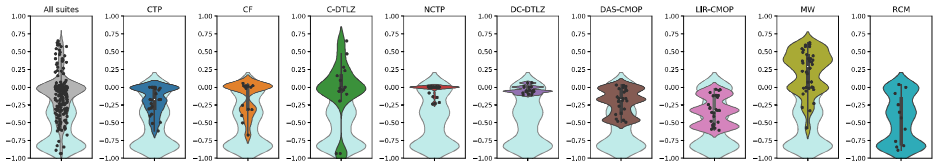
- Assess the existing test suites of CMOPs
- Measure **correlations between objectives and constraints**
- Identify feasible **subregions and basins**
- Characterize the **local structure** of violation landscapes

Methods:

- Space-filling sampling
- Random and adaptive walk
- Information content

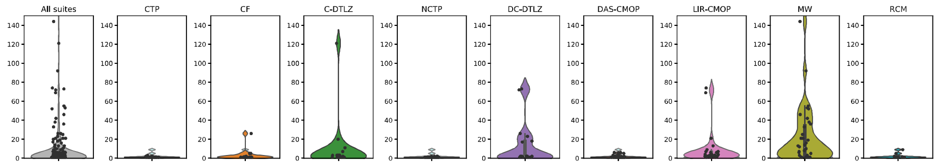
Test Suite Comparison: Correlations

Correlations between objectives and constraints



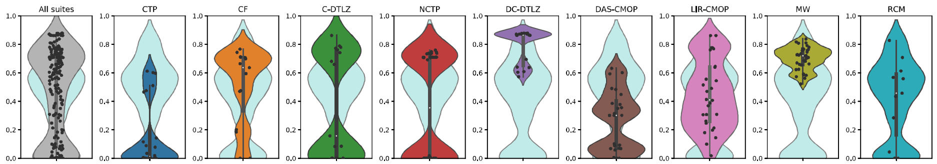
Test Suite Comparison: Feasible Subregions

Number of feasible subregions



Test Suite Comparison: Ruggedness

Maximum information content



A Note on Present Test Suites

- There are too **many Type I and Type II** CMOPs in the existing suites (Tanabe and Oyama, 2017)
- Pareto front shapes of the artificial test problems are **unrealistically regular** (Ishibuchi et al., 2019)
- The existing artificial test problems **fail to satisfactorily represent some real-world problem characteristics** (Vodopija et al., 2022)
- The predominant source of complexity in test CMOPs lies solely in the **Pareto front approximation** (Vodopija et al., 2024)

Software for Constrained Multiobjective Optimization

Overview

- Python
- R
- Matlab
- Java

Python (i)

pymoo: Multi-objective Optimization in Python[5]

- Algorithm implementations: NSGA-II (CDP), NSGA-III (CDP), C-TAEA
- CMOP implementations: TNK, OSY, BNH, CTP, DAS-CMOP, MW, MODAct, 3 RCMs
- Performance assessment: HV, GD, GD^+, IGD, IGD^+
- Additional: Solution repair when constraints are analytically expressed and visualization techniques

[5] https://pypi.org/project/pymoo/

Python (ii)

jMetalPy: Python Version of the JMetal Framework[6]

- Algorithm implementations: NSGA-II (CDP), NSGA-III (CDP), MOEAD-IEpsilon
- CMOP implementations: SRN, TNK, OSY, BNH, LIR-CMOP
- Performance assessment: HV, GD, IGD, EPS
- Additional: Statistical analysis and visualization techniques

[6] https://pypi.org/project/jmetalpy/

Python (iii)

deap: Distributed Evolutionary Algorithms in Python[7]

- Constraint handling by delta penalty approach, closest valid penalty approach, or island approach

pygmo: Parallel Optimization for Python[8]

- Advanced algorithms for hypervolume calculation

[7] https://pypi.org/project/deap/
[8] https://pypi.org/project/pygmo/

R

mco: Multiple Criteria Optimization Algorithms and Related Functions[9]

- Algorithm implementations: NSGA-II (CDP)
- CMOP implementations: BNH
- Performance assessment: HV, GD, EPS

MOEADr: Component-Wise MOEA/D Implementation[10]

- Constraint handling by penalty function approach, violation-based ranking
- Performance assessment: HV, IGD

[9] https://cran.r-project.org/web/packages/mco/index.html
[10] https://cran.r-project.org/web/packages/MOEADr/index.html

Matlab

PlatEMO: Evolutionary Multi-objective Optimization Platform[11]

- Algorithm implementations: NSGA-II (CDP), NSGA-III (CDP), ToP, PPS-MOEA/D, C-TAEA, MOEA/D-DAE, CCMO, etc.
- CMOP implementations: CF, C-DTLZ, DC-DTLZ, LIR-CMOP, MW, DOC, ZXH-CF, etc.
- Performance assessment: HV, GD, IGD, IGD+
- Additional: Statistical analysis, visualization techniques and GUI

[11] https://github.com/BIMK/PlatEMO

Java

MOEAFramework: A Free and Open Source Java Framework for Multiobjective Optimization[12]

- Algorithm implementations: NSGA-II (CDP)
- CMOP implementations: SRN, TNK, OSY, BNH, CF, C-DTLZ
- Performance assessment: HV, GD, IGD
- Additional: Statistical analysis and visualization techniques

jMetal: A Framework for Multi-objective Optimization with Metaheuristics[13]

- Same functionalities as jMetalPy

[12] https://github.com/MOEAFramework/MOEAFramework
[13] https://github.com/jMetal/jMetal

Conclusions

Summary

- Increasing interest in **constrained multiobjective optimization**
- Many new **techniques, test suites, and software** proposed in the last years
- **Problem characterization** is now gaining interest

Open Issues and Future Research Directions

- Advances in **exploratory landscape analysis** for CMOPs
- Artificial **test suites** reflecting real-world problem characteristics
- Comprehensive **algorithm performance assessment** in solving CMOPs
- **Assessment of the recently proposed CHTs** on real-world problems

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